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INTEGRAL REPRESENTATIONS OF FINITE TRANSFORMATION GROUPS III: SIMPLY-CONNECTED FOUR-MANIFOLDS

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The action of a finite group G on a simply-connected 4-manifold X yields a $\mathbb{Z}G$-lattice $H_2(X)$. In this paper we study the relations between the geometry of the G-action on X and the homological and representation theoretic properties of $H_2(X)$.

Introduction

Let G be a finite group, X a topological G-space (possibly with additional geometric structure) and $R=\mathbb{Z}$, or an algebraically closed field of characteristic dividing the order of G, and $\mathscr{F}$ a compatible coefficient system (e.g. a G-sheaf over X) of R-modules. Then the induced G-action on $H_*(X;\mathscr{F})$ gives them an RG-module structure. A general question arises as to how the G-module structure of $H^*(X;\mathscr{F})$ or $H_*(X;\mathscr{F})$ are related to the topology of X (and the additional geometric structure). This is a general formulation of problems which are widely studied in several areas of mathematics, including topology, algebraic and arithmetic geometry, number theory, and representation theory. We are interested in the topological and homological-algebraic aspects of this problem in the special circumstances arising from the consideration of symmetries of manifolds. In our previous papers [1–8], we have studied certain aspects of such problems from the point of view of group cohomology and algebro-geometric invariants of the equivariant cohomology of X. In this paper and its sequal [10] we specialize to study the G-module structure of $H_2(X)$ when X is a simply-connected manifold and its relationship to the topology of the G-action on X. It turns out that only certain homological and regularity

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aspects of these G-manifolds are needed at this level. Therefore we have formulated our results in the context of Poincaré complexes with additional structure. This is also intended to provide the algebraic-topological framework for construction and classification of finite symmetries of simply-connected 4-manifolds and varieties which will be treated elsewhere.

The main results of this paper are in Sections 4, 5, and 6. In Section 4, we show that for $G=\mathbb{Z}_p$, $H_2(X)$ and the singular set of the action determine each other (Theorems 4.8, 4.9, 4.14). In Section 5, we apply this information to determine the G-module $H_2(X)$ for more general finite groups G. In Section 6, we present a necessary and sufficient condition for an infinite-dimensional free G-complex Y to be G-homotopy equivalent to a finitely dominated Poincaré G-complex (Theorem 6.1). We found it useful to include a section on finite domination and finite dimensionality as these results will be needed in Section 6, and some of the statements are scattered in the literature.

A few words to justify the group-cohomology approach of this paper versus other more traditional representation theoretic methods are in order. As it stands at present, there are no classification theorems for general RG-modules except for very special groups such as cyclic G. For instance, when $G \supseteq \mathbb{Z}_p \times \mathbb{Z}_p$ for $p \geq 3$, RG has wild representation type and classification of RG-modules is considered a hopeless task, cf. [14]. Thus, in order to have a chance to determine the structure or invariants of our modules, we need strong restrictions imposed on them by the topology of X. In the present paper, we proceed to supply such restrictions by studying the G-chain complexes arising from the topology of X. It turns out that a great deal of cohomological information becomes available from this point of view, which leads surprisingly to complete answers in favorable circumstances. Moreover, these restrictions point out the potential answers in more general circumstances. In a forthcoming paper, we apply these to the cases where X is endowed with more geometric structure, e.g. an algebraic variety.

The organization of this paper is as follows. In Section 1, we review some standard algebraic background while introducing a few notations and terminologies.

In Section 2, we discuss the construction of G-modules (up to stability) from cohomology classes, and a particularly useful class of modules is briefly studied. The construction and early applications of similar modules in the context of modular representation theory are due to Carlson [12], while the ω-operators used in Section 2 are the suitably stabilized Heller and Eckmann–Hilton operators [17, 18]. In Section 3, we collect some facts on finite domination. Section 4 studies some general questions which arise in conjunction with the G-module structure of $H_2(X)$ and the singular set of the G-action on a simply-connected 4-manifold in the context of regular Poincaré complexes. In particular, these lead to a complete determination of $H_2(X)$ in terms of the topology of the action when $G=\mathbb{Z}_p$. When G acts freely on X and G has periodic cohomology of period 4, Hambleton and Kreck [16] have also determined (independently and earlier) the structure of $H_2(X)$ in a different context. Thus our Proposition 4.3 for $G=\mathbb{Z}_p$ and Proposition 4.4 for general G

overlap with [16] somewhat. Since the formulation and the approach in Proposition 4.4 is different, we have seen no harm in including both results and referring the reader to [16] for an alternate description. Section 5 has basically the local-to-global arguments which determine the G-module structure of $H_2(X)$ from its restriction to the prime order subgroups of G. In Section 6, we apply the methods and results of Section 4 to study the finite domination and the finiteness obstructions which arise in constructing G-actions on 4-manifolds. Finally, let us point out some papers of related interest: [15, 16, 23].

1. Notation and preliminaries

Throughout this paper, G denotes a finite group, and p a prime divisor of $|G| \equiv$ order of G. R denotes a commutative ring with 1, although we will restrict our attention to $R=\mathbb{Z}$ or R = field in the statements of our results for the most part. The group ring RG, is a quasi-Frobenius R-algebra (cf. [12]). For example, this fact is used only to the extent that implies RG-projectives and RG-injectives are the same.

Let $\mathcal{M}(G)$ be the category of R-free finitely generated RG-modules. The set of isomorphism classes of the objects of these categories form abelian monoids under direct sum of modules, with the trivial module as the zero element. Introduce the Grothendieck construction by the following equivalence relation: consider all short exact sequences $0 \to A \to C \to B \to 0$, and declare $[A] \oplus [B]$ equivalent to $[C]$. The set of equivalence classes becomes an abelian group in each case. They are denoted by $K_0(RG)$ and $G_0(RG)$ respectively. The equivalence classes of free RG-modules form subgroups of $K_0(RG)$ and $G_0(RG)$ and their quotient groups are $\tilde{K}_0(RG)$ and $\tilde{G}_0(RG)$ respectively. The forgetful functor $\mathscr{P}(RG) \to \mathcal{M}(RG)$ induces homomorphisms $K_0(RG) \to G_0(RG)$ and $\tilde{K}_0(RG) \to \tilde{G}_0(RG)$ which are called the Cartan homomorphism. In general, the Cartan homomorphism is neither injective nor surjective. However, for $G=\mathbb{Z}_p \equiv \mathbb{Z}/p\mathbb{Z}$, it is injective. We will use this fact in the sequel. The reader may consult [21] or [14] for details.

The Tate cohomology groups of an RG-module M are denoted by $\hat{H}^i(G,M)$, $i \in \mathbb{Z}$. For $i>0$, $\hat{H}^i(G,M)$ is the same as ordinary cohomology groups $H^i(G,M)$, and for $i<0$, $\hat{H}^{i-1}(G,M) \cong H_i(G,M)$. We refer the reader to [13, Chapter XII] for a discussion of these groups as well as the ordinary cohomology groups.

The modules $R(G/H)$ with the natural action of G, $H \subseteq G$, are referred to as permutation modules. Let C_* be a chain complex of RG-modules. We call C_* a permutation complex if the following two conditions are satisfied:

(PC1) Each C_i is a permutation module whose R-basis is a G-set S_i;

(PC2) The boundary homomorphisms $\partial_{i+1}: C_{i+1} \to C_i$ satisfy $\partial_{i+1}(S_{i+1}^H) \subset R(S_i^H) \subset C_i^H$, where $R(S_i^H)$ is the free R-module generated by the set of H-fixed points S_i^H.

The reader is referred to [1] for further properties and constructions of permutation complexes. It should be noted that in the literature, the term permutation complex is used often for an RG-chain complex which satisfies (PC1) only. It is easy to see that most RG-chain complexes which satisfy (PC1) do not satisfy (PC2) necessarily. Therefore, our permutation complexes are much more restricted. If X is a G-CW complex, then the cellular chains of X, say $C_*(X) \otimes R$, is a permutation RG-complex. If G acts cellularly (respectively simplicialy) on a CW complex X (respectively simplicial complex X), then after subdivisions (two barycentric subdivisions), X becomes a G-CW complex. In particular, if G acts smoothly on a manifold X, then X admits a G-CW complex structure.

Let E_G be a contractible free G-space. Then the equivariant homology and cohomology of a G-space X are defined by $H^i_G(X; R) \equiv H^i(E_G \times_G X; R)$ and $H_i^G(X; R) \equiv H_i(E_G \times_G X; R)$, where $E_G \times_G X$ is the twisted product. For a discussion of equivariant cohomology, in particular, the localization theorem, we refer to [19], and [8] for an algebraic analogue.

Finally, the singular set of a G-space is the union of all the fixed point sets of nontrivial subgroups of G.

2. Modules, cohomology, and stability

In this section we describe some algebraic constructions based on stable versions of the loop and suspension operators of Heller and Eckmann and Hilton. We also need to generalize Carlson's construction of modular representations from group cohomology classes, in order to obtain the stable version which is more suitable for our purposes. Further details and related constructions are provided in [9].

2.1. Definition. Let M_1 and M_2 be two RG-modules. A projective stable equivalence consists of the following commutative diagram, and we denote it by $M_1 \sim M_2$ for short:

$$\begin{tikzcd} M_1 \oplus P_1 \arrow[r, "\cong", "g"'] & M_2 \oplus P_2 \arrow[d, "\pi"] \\ M_1 \arrow[u, "j"] \arrow[r, "h"] & M_2 \end{tikzcd}$$

Here, j and π are the canonical injection and projection, g is an RG-isomorphism, and P_1 and P_2 are finitely generated RG-projective RG-modules. If in the above, P_1 and P_2 are free RG-modules, then we call M_1 and M_2 *stably RG-isomorphic*, and we denote it by $M_1 \overset{f}{\sim} M_2$. If we need to emphasize G, the notation will be $M_1 \underset{G}{\sim} M_2$, etc.

2.2. Lemma. *A projective stable equivalence $h: M_1 \sim M_2$ induces a canonical isomorphism $h_*: \hat{H}^i(K, M_1) \cong \hat{H}^i(K, M_2)$ for all $K \subseteq G$, and all $i \in \mathbb{Z}$.*

Proof. RG-projective modules are cohomologically trivial. Hence we have $j_* =$ identity, $\pi_* =$ identity, and g_* is an isomorphism. □

2.3. Proposition. *Suppose M is an R-free RG-module. Let* $\mathrm{Hom}_G^{\sim}(M,R) = \{f: M \to R \mid f$ *is an RG-homomorphism which factors through an RG-projective module*$\}$. *Then*

$$\mathrm{Hom}_G(M,R)/\mathrm{Hom}_G^{\sim}(M,R) \cong \hat{H}^0(G, \mathrm{Hom}_R(M,R)).$$

Proof. Let $\nu_G = \sum_{g\in G} g \in RG$ be the norm element. Then $\mathrm{Hom}_R(M,R)^G \cong \mathrm{Hom}_G(M,R)$, and $\nu_G \cdot \mathrm{Hom}_R(M,R) \cong \mathrm{Hom}_G^{\sim}(M,R)$. See [13, Chapter XII]. □

(Compare also [20, Theorem 3.6, pp. 74–75]).

2.4. Definition. If $\varphi: M \to R$ is an RG-homomorphism we denote by $\tilde{\varphi} \in \hat{H}^0(G, \mathrm{Hom}_R(M,R)) \equiv \widehat{\mathrm{Hom}}_G(M,R)$ its equivalence class. If φ is surjective, we denote $\mathrm{Ker}\,\varphi$ by L_φ. Let M^* denote $\mathrm{Hom}_R(M,R)$ for short.

In the following, assume that all RG-modules are R-free.

2.5. Proposition. *Let $\varphi_1, \varphi_2: M \to R$ be two surjective RG-homomorphisms such that $\tilde{\varphi}_1 = \tilde{\varphi}_2$. Then $L_{\varphi_1} \sim L_{\varphi_2}$.* □

2.6. Proposition. (a) *Suppose $h: M_1 \sim M_2$, $\varphi_1: M_1 \twoheadrightarrow R$, $\varphi_2: M_2 \twoheadrightarrow R$ are two RG-homomorphisms such that $h_*(\tilde{\varphi}_1) = \tilde{\varphi}_2$. Then $L_{\varphi_1} \sim L_{\varphi_2}$.*

(b) *Suppose $\varphi_i: M_i \twoheadrightarrow R$, $i = 1, 2$, are two RG-homomorphisms such that $L_{\varphi_1} \sim L_{\varphi_2}$. Then there exists a projective stable equivalence $f: M_1 \sim M_2$ such that $f_*(\tilde{\varphi}_1) = \tilde{\varphi}_2$.* □

We define another notion of stable equivalence based on taking syzygies. Define $\omega(M)$ for any finitely generated RG-module M via the following exact sequence:

$$0 \to \omega(M) \to (RG)^m \to M \to 0.$$

$\omega(M)$ is well-defined up to RG-projective stable equivalence, by the Schanuel Lemma [21]. We may also use an RG-projective module instead of $(RG)^m$. Let $\omega^1(M) = \omega(M)$, $\omega^0(M) = M$ and $\omega^{n+1}(M) = \omega(\omega^n(M))$ inductively, $n \in \mathbb{Z}$.

2.7. Proposition. $\widehat{\mathrm{Hom}}_G(\omega^n(M), R) \cong \hat{H}^n(G, M^*)$, *where* $M^* = \mathrm{Hom}_R(M,R)$ *with 'the diagonal' RG-module structure.* □

(See [20, pp. 74–75] also.)

2.8. Corollary. *Let $\zeta \in \hat{H}^n(G, M^*)$ be represented by the surjective RG-homomorphism $\zeta: \omega^n(M) \to R$. Then* $\mathrm{Ker}\,\zeta = L_\zeta$ *is well-defined up to projective stable equi-*

valence. If we use only free modules in the definition of ω, then L_ζ is determined up to stable equivalence. □

2.9. Proposition. (a) *Let $\varphi : M \to R$ be a surjective RG-homomorphism, and $\tilde{\varphi} \in \hat{H}^0(G, M^*)$ the corresponding cohomology class, and $\omega^n(\varphi) \in \widehat{\mathrm{Hom}}_G(\omega^n(M), \omega^n(R))$ the class of the induced RG-homomorphism in*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \omega^n(M) & \longrightarrow & F_n & \longrightarrow \cdots \longrightarrow & F_1 & \longrightarrow & M & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \omega^n\varphi} & & & & \Big\downarrow & & \Big\downarrow{\scriptstyle\varphi} & & \\ 0 & \longrightarrow & \omega^n(R) & \longrightarrow & F_n' & \longrightarrow \cdots \longrightarrow & F_1' & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

where F_i and F_i' are RG-free. Then we may choose representatives so that $\omega^n\varphi$ is surjective.

(b) $\mathrm{Ker}(\omega^n\varphi) \sim \omega^n \mathrm{Ker}\,\varphi$. *Hence $L_{\omega^n\varphi} \sim \omega^n L_\varphi$.* □

2.10. Remark. (a) The notation L_ζ is chosen to suggest the similarity between the construction of such modules in this section and those of Carlson [12] which may be considered their unstable predecessors.

(b) The operations $\Omega^n(-)$, $n \in \mathbb{Z}$ have been introduced by Heller [17] and Eckmann and Hilton [18], using minimal projective resolutions and projective covers whenever they exist (e.g. R = field or p-adic integers). Again, we need only a stable analogue, which does not rely on the existence of projective covers or minimal resolutions. Thus, the notation ω is used to differentiate between the stable and unstable operators.

We comment briefly on how the cohomological information leads to more precise information on the level of modules. The following is perhaps well-known to the experts.

2.11. Proposition. *Let M_1 and M_2 be R-free finitely generated RG-modules, and let $f : M_1 \to M_2$ be an RG homomorphism which induces an isomorphism on the level of Tate cohomology, i.e. for each subgroup $H \subseteq G$, $f_* : \hat{H}^*(H, M_1) \xrightarrow{\cong} \hat{H}^*(H, M_2)$. Then M_1 and M_2 belong to the same projective stable equivalence class. Thus $M_1 \oplus P_1 \cong M_2 \oplus P_2$ for suitable RG-projectives P_1 and P_2.*

Proof. Cf. [5] or [2]. We need only prime power order subgroups $H \subseteq G$ in the above, if $R = \mathbb{Z}$, and only p^j-order subgroups, if R is a field of characteristic p. □

In the situations where the Krull–Schmidt–Azumaya theorem applies [13], Proposition 2.11 implies that the non-projective indecomposable RG-summands of M_1 and M_2 are isomorphic. In particular, if M_1 and M_2 have no projective summands, then $M_1 \cong M_2$. This is formalized in the following:

2.12. Corollary. *Suppose the Krull–Schmidt–Azumaya theorem holds for RG, and that M_1 and M_2 have no indecomposable RG-projective summands. Then in Proposition* 2.11 *above, M_1 and M_2 are RG-isomorphic.* □

2.13. Remark. The cohomology of the modules L_ζ may be computed from the short exact sequence $0 \to L_\zeta \to \omega^n(M) \to R \to 0$ and the associated long exact sequence:

$$\cdots \longrightarrow \hat{H}^j(G, L_\zeta) \longrightarrow \hat{H}^j(G, \omega^n(M)) \xrightarrow{\zeta_*} \hat{H}^j(G, R) \longrightarrow \cdots$$

in which ζ_* can be identified with a suitable cup product operation by the corresponding cohomology class $\tilde{\zeta} \in \hat{H}^r(G, M^*)$. For example, let $n = 2r > 0$ and suppose that $\tilde{\zeta} \in \hat{H}^{2r}(G, R)$ is a non-nilpotent element, and $G \cong (\mathbb{Z}_p)^m$ and R is a field of characteristic p. Then the following diagram commutes:

$$\begin{tikzcd} \hat{H}^{j+2r}(G, \omega^{2r}(R)) \arrow[rr, "\zeta_*"] & & \hat{H}^{j+2r}(G, R) \\ & H^j(G, R) \arrow[ul, "\delta", "\cong"'] \arrow[ur, "\cdot\tilde{\zeta}"'] & \end{tikzcd}$$

where δ is the usual isomorphism induced from shifting, and $\cdot\,\tilde{\zeta}$ is cup product with the class $\tilde{\zeta}\cdot$. In this set-up, $\cdot\,\tilde{\zeta}$ is injective, therefore, for p odd and in sufficiently large dimensions,

$$H^*(G, L_\zeta) \cong \frac{R[t_1, \ldots, t_m] \otimes \Lambda(u_1, \ldots, u_m)}{(\tilde{\zeta})}$$

where $u_i \in H^1(G, R)$, and $t_i \in H^2(G, R)$ are the Bocksteins of u_i, and $\tilde{\zeta}$ is a homogeneous polynomial of degree $r \in R[t_1, \ldots, t_m]$. If $p = 2$, then $H^*(G, L_\zeta) \cong R[u_1, \ldots, u_m]/(\tilde{\zeta})$ where $\tilde{\zeta}$ is a homogeneous polynomial of degree r in the u_i's. In the latter case, r need not be even. One final comment in this direction is that in Proposition 2.11 above, we may assume that f_* induces isomorphisms in all sufficiently high dimension, so that computations such as the above are quite useful in this respect.

3. Finite dimensionality and finite domination

In this section we collect some facts which we will need in Section 6. The main references are the papers of Wall [22] (cf. also [9]). According to an old result, attributed to J. Moore, a simply-connected topological space which is homotopy equivalent to a CW complex with finitely generated homology, has in fact the homotopy-type of a finite complex. If G is a finite group, and $\pi_1(X) = G$, then one has the following generalization: Suppose $H_*(X; \mathbb{Z}\pi_1(X)) \cong H_*(\tilde{X}; \mathbb{Z})$ (where $\tilde{X}$ is the universal covering space of X) is finitely generated and $\dim(X) < \infty$, then X is finitely dominated. In particular, X is homotopy equivalent to a CW complex of finite-type (i.e. with finitely many cells in each dimension). The proof uses the fact that $\mathbb{Z}\pi_1(X)=\mathbb{Z}G$ is Noetherian since G is finite, and the arguments of [22] to construct an n-connected map $\psi: L \to X$, where L is a finite complex and $\dim X<n$. Then Lemma 3.1 of [22, I] shows that ψ has a homotopy right inverse, hence L dominates X. Under such circumstances, there is a well-defined element $\sigma(X) \in \tilde{K}_0(\mathbb{Z}G)$ represented by the finitely generated projective $\mathbb{Z}G$-module $\pi_{n+1}(\psi)$, with the property that $\sigma(X)=0$ if and only if X is homotopy equivalent to a finite complex.

Thus, to study finite domination of CW complexes W with $\pi_1(X) \equiv G \equiv$ finite group, and finitely generated homology, the problem is reduced to showing that $\dim X<\infty$. At this point, the problem can be formulated in terms of chain complexes. Namely, according to [22, II, Theorem 2 and Corollary 2.1] finite domination and finite dimensionality of a space X (up to homotopy) is a property of the $\mathbb{Z}\pi_1(X)$-free chain complex of its universal covering space (up to chain homotopy). On the other hand, Wall has the following finite dimensionality criterion:

3.1. Theorem (Wall [22, II, Theorem 6]). *A positive projective chain complex C_* is chain homotopy equivalent to an n-dimensional complex if and only if $H_i(C_*)=0$ for $i>n$ and* $\operatorname{Im}(d: C_{n+1} \to C_n)$ *is a projective module.* □

According to the above, we need to study $\mathbb{Z}G$-projectivity of the module $\operatorname{Ker}(d_n)$ for some sufficiently large n. As in the above discussion, we may also assume that C_* is finitely generated in each dimension, so that $\operatorname{Ker}(d_n)$ is $\mathbb{Z}G$-finitely generated. Let $k=\bar{\mathbb{F}}_p=$ an algebraic closure of $\mathbb{F}_p$. According to [8, Proposition 1.1 and Corollary 1.3] the finite domination of C_* reduces to showing that for each $p \mid |G|$ and each shifted cyclic subgroup $S \subset kG$, the kS-free chain complex $C_* \otimes k$ is finitely dominated (as a kS-chain complex). The projectivity criterion of [3] shows the following:

3.2. Theorem. *If $C_*=C_*(X; \mathbb{Z}\pi_1(X))$ in Theorem* 3.1 *above, then* $\operatorname{Ker}(d_n)$ *is $\mathbb{Z}G$-projective if and only if it is $\mathbb{Z}C$-projective for each cyclic subgroup $C \subseteq G$ of prime order.* □

Putting together the above discussion, we have the following local-to-global criterion which will be used in Section 6.

3.3. Corollary. *Suppose that $\tilde{X}$ is a simply-connected free G-space and let $X=\tilde{X}/G$. Then X is finitely dominated if and only if $\tilde{X}/C$ is finitely dominated for each prime order subgroup $C \subseteq G$.* □

4. Regular group actions in dimension four

The main results of this section determine the relationship between the singular set of a regular G-action on X and the G-module structure of $H_2(X)$ up to projective or free equivalence. The emphasis in this section is mostly on the level of cyclic subgroups of G. In the following section we show how this information about cyclic subgroups determines $H_2(X)$ as a G-module.

4.1. Definition. Let X be a connected oriented Poincaré complex of formal dimension four, and let G be a finite group action on X. This action is called *regular* if all elements $g \in G$ act via orientation preserving cellular homeomorphisms such that their fixed point sets are disjoint unions of a finite number of points and closed Riemann surfaces. Without loss of generality, we may assume also that X becomes a G-CW complex under this action. X is called then a regular Poincaré G-complex. As before, $H_*(X)$ is finitely generated.

Notice that any smooth G-manifold of dimension 4 has a regular action. More generally, if X is a PL-manifold and elements of G act by simplicial maps for some simplicial structure on X, then again the action is regular. The manifold condition is not in fact necessary if $\dim X = 4$ and X satisfies Poincaré duality and the action is simplicial. The proofs of these facts are standard under such simplifying assumptions.

The G-action on X induces an RG-module structure on $H_2(X; R)$. We are interested in studying this RG-module. Throughout this section, the standing hypotheses will be the following:

(SPC) X is a finite Poincaré complex of dimension four with a regular G-action, and $\pi_1(X) = 0$.

Some of these conditions are not necessary for all the arguments, and they are assumed only for convenience. For example, in most of the arguments involving only the cellular chains, we only need $H_1(X) = 0$. Often X need have only finite dimension with finitely-generated homology, if we confine ourselves to projective stable equivalence. The following lemma then justifies the extra conditions:

4.2. Lemma. *Suppose X is a regular Poincaré G-complex as in Definition* 4.1, *and such that* $\dim X < \infty$. *Then X is G-homotopy equivalent to a* 4*-dimensional regular Poincaré G-complex.*

Proof. Let $\dim X = n > 4$. Then the exact sequence

$$0 \longrightarrow C_n(X) \longrightarrow \cdots \longrightarrow C_5(X) \xrightarrow{\partial_5} B_4(X) \longrightarrow 0$$

shows that $B_4(X)$ is stably free over $\mathbb{Z}G$, since $C_i(X)$ are $\mathbb{Z}G$-free for $i \geq 3$. Let X_4 be some G-invariant 4-skeleton for X. Then we have the exact sequence: $0 \to B_4(X) \to H_4(X_4) \to H_4(X) \to 0$ in which all $\mathbb{Z}G$-modules are $\mathbb{Z}$-free. This sequence splits, therefore, over $\mathbb{Z}G$, and $H_4(X_4) \cong H_4(X) \oplus B_4(X)$. Without loss of generality, we may arrange for $B_4(X)$ to be $\mathbb{Z}G$-free. The injection $j: B_4(X) \to C_4(X_4)$ splits as a $\mathbb{Z}G$-direct summand, and we let $k: C_4(X_4) \to B_4(X)$ be the corresponding projection. Since $C_3(X_4)$ is $\mathbb{Z}G$-free, relative to $(C_i(X_4), i \le 2)$, we can find a G-chain homotopy equivalence from $C_*(X_4)$ to C'_* below: $C'_3 = C_3(X_4) \oplus B_4(X)$ and $C'_4 = C_4(X_4)$; $\partial' : C'_3 \to C'_2 = C_2(X)$ is ∂_3 on $C_3(X_4)$ and zero on $B_4(X)$; $\partial'_4 : C'_4 \to C'_3$ is $\partial_4 \oplus k : C_4(X_4) \to C_3(X_4) \oplus B_4(X)$. Using the Hurewicz theorem, as in [21], we may geometrically realize C'_*, since the problem involves only free 3-cells and 4-cells. The resulting G-CW complex $X' \supset X_2$ is G-homotopy equivalent (relative to the 2-skeleton X_2) to X_4 and $\dim X' = 4$. □

In the following, we relate the structure of the singular set of the action and the G-module $H_2(X)$. Without further mention all actions are regular and they satisfy (SPC) above. (Compare Propositions 4.3 and 4.4 with [16, Proposition 2.4].)

4.3. Proposition. *Suppose G acts freely on X, $G = \mathbb{Z}_p$. Then $H_2(X) \cong I_G \oplus I_G \oplus F$, where F is a free $\mathbb{Z}G$-module.*

This is a special case of the following:

4.4. Proposition. *Let G act freely on X. Then,*

(a) *there exists a $\mathbb{Z}G$-module K such that $H^2(X) = \omega^2(K)$ and K is obtained from the extension (η) below:*

$$0 \longrightarrow K \longrightarrow \omega^{-5}(\mathbb{Z}) \xrightarrow{\varphi} \mathbb{Z} \longrightarrow 0; \qquad (\eta)$$

(b) *up to projective stable equivalence, the extension (η) is determined by a class $e(\eta) \in H_4(G; \mathbb{Z})$;*

(c) *for any $u \in H_4(G; \mathbb{Z})$, there exists a smooth closed G-manifold X,* $\dim X = 4$, *$\pi_1(X) = 0$, and $H_2(X)$ satisfies an extension (η) as in* (a) *such that the associated homology class $e(\eta)$ of* (b) *is the given class u.*

We give an independent proof of Proposition 4.3 first:

Proof of Proposition 4.3. Since $G = \mathbb{Z}_p$ has periodic cohomology, $\omega^2(\mathbb{Z}) \sim \mathbb{Z}$ and $\omega^{-2}(\mathbb{Z}) \sim \mathbb{Z}$. We compute Euler–Poincaré characteristics in $\tilde{G}_0(\mathbb{Z}G)$ and then we identify the ambiguous projective $\mathbb{Z}G$-module factor via the injectivity of the Cartan map $\tilde{K}_0(\mathbb{Z}G) \to \tilde{G}_0(\mathbb{Z}G)$. In $\tilde{G}_0(\mathbb{Z}G)$ we have

$$[\chi H_*(X)] = [\chi C_*(X)] = 0,$$

thus $2[\mathbb{Z}] + [H_2(X)] = 0$, since $C_*(X)$ is $\mathbb{Z}G$-free. We have the following exact sequence in which F_0 is a free $\mathbb{Z}G$-module:

$$0 \to H_2(X) \to F_0 \to \mathbb{Z} \oplus \mathbb{Z} \to 0.$$

Comparing this with the sequence

$$0 \to I_G \oplus I_G \to (\mathbb{Z}G)^2 \to \mathbb{Z} \oplus \mathbb{Z} \to 0$$

and applying Schanuel's lemma [21], we conclude:

$$H_2(X)\oplus(\mathbb{Z}G)^2\cong I_G\oplus I_G\oplus F_0.$$

From the classification of $\mathbb{Z}[\mathbb{Z}_p]$-modules [14], it follows that $H_2(X)\cong I_G\oplus I_G\oplus F$, where F is a free $\mathbb{Z}G$-module. $\square$

Proof of Proposition 4.4. Consider the free $\mathbb{Z}G$-chain complex $C_*(X)$: We have the following exact sequences in which $Z_i(\cdot)$ and $B_i(\cdot)$ indicate cycles and boundaries:

$$0\to\mathbb{Z}\to C_4(X)\to C_3X\to B_2(X)\to 0,$$

$$0\to B_2(X)\to Z_2(X)\to H_2(X)\to 0,$$

$$0\to Z_2(X)\to C_2(X)\to C_1(X)\to C_0(X)\to\mathbb{Z}\to 0.$$

(E) Denoting duals (i.e. $\mathrm{Hom}_{\mathbb{Z}}(\cdot,\mathbb{Z})$) by an upper asterisk, we have:

$$0\to H^2(X)\to Z_2(X)^*\to B_2(X)^*\to 0.$$

From (E) above, $B_2(X)^*\sim\omega^2(\mathbb{Z})$, and $Z_2(X)^*\sim\omega^3(\mathbb{Z})^*\sim\omega^{-3}(\mathbb{Z})$. Therefore, up to stable equivalence we have the following exact sequence:

$$0\to H^2(X)\to\omega^{-3}(\mathbb{Z})\to\omega^2(\mathbb{Z})\to 0.$$

From Section 2, the stable class of $H^2(X)$ is determined from:

$$0\longrightarrow\omega^{-2}H^2(X)\longrightarrow\omega^{-5}(\mathbb{Z})\xrightarrow{\varphi}\mathbb{Z}\longrightarrow 0.$$

The equivalence class of φ in $\widehat{\mathrm{Hom}}_G(\omega^{-5}(\mathbb{Z}),\mathbb{Z})$ determines $\omega^{-2}H^2(X)$ up to projective stable equivalence. Since $\widehat{\mathrm{Hom}}_G(\omega^{-5}(\mathbb{Z}),\mathbb{Z})\cong\hat{H}^0(G,\mathrm{Hom}_{\mathbb{Z}}(\omega^{-5}(\mathbb{Z}),\mathbb{Z}))\cong\hat{H}^{-5}(G;\mathbb{Z})\cong H_4(G;\mathbb{Z})$, the class of φ corresponds $\varepsilon(\eta)\in H_4(G;\mathbb{Z})$. Moreover, $\omega^{-2}H^2(X)\sim L_\varphi$, so that $H_2(X)\cong H^2(X)^*\sim(\omega^2L_\varphi)^*\sim\omega^{-2}(L_\varphi^*)$. Since $e(\eta)$ determines $H_2(X)$ up to projective stable equivalence (see Section 2 above) and the extension (η), we have proved (a) and (b) above.

To prove (c), we consider the homomorphism $\varrho:\tilde{\Omega}_4(BG)\to\tilde{H}_4(BG)=H_4(G)$ from the oriented bordism group to the homology group given by $\varrho(f:M^4\to BG)=f_*[M^4]\in H_4(BG)$ the image of the fundamental (orientation) class of M. Consider the Atiyah–Hirzebruch spectral sequence: $H_i(BG;\Omega_j(\text{point}))\Rightarrow\mathfrak{G}r(\Omega_{i+j}(BG))$. The relevant coefficient groups are $\Omega_0=\mathbb{Z}=\Omega_4$ and $\Omega_i=0$, $i=1,2,3$. Thus we have $\omega_4(BG)\cong H_4(BG)\oplus\Omega_4(\text{point})$, so that ϱ is surjective. Let $u\in H_4(G;\mathbb{Z})\cong H_4(BG)$ be given, and choose a representative $f:W^4\to BG$ for a class $u'\in\tilde{\Omega}_4(BG)$ such that $\varrho(u')=u$. Since W is oriented, we can do surgery on the map f to adjust the fundamental group of W, and assume that the representative (W^4,f) is chosen so that $f_*:\pi_1(W)\to\pi_1(BG)$ is an isomorphism. Let X be the universal covering space of W. Then X is a smooth, simply-connected, closed free G-manifold, for which the associated homology class $e(\eta)=u$.

See [10] for more details and a generalization of this result. $\square$

4.5. Corollary. *If G acts trivially on $H_2(X)$, then the G-action on X is not free. In particular, if $G=\mathbb{Z}_p$, then $X^G\neq\emptyset$.*

Next, we will relate the projective stable equivalence class of the $\mathbb{Z}G$-module $H_2(X)$ to the singular set of the G-action on X. First, there is a convenient cohomological condition for projective stable equivalence which shows that in many cases cohomological computations for $\mathbb{Z}_p$-subgroups of G is sufficient. See Section 5 for another instance of this.

4.6. Proposition. *Let X^4 and Y^4 satisfy (SPC) above.*

(a) *Let $f: Y\to X$ be a G-map, and consider $f^*: H^j(C; H^i(X))\to H^j(C; H^i(Y))$ induced by $f^*: H^*(X)\to H^*(Y)$. Assume that f^* is an isomorphism for all i, all $j\geqslant 0$, and all prime order subgroups $C\subseteq G$. Then $H_2(X)\sim H_2(Y)$.*

(b) *Under the hypothesis of* (a), *$f^K: Y^K\to X^K$ is a homotopy equivalence. In particular, if $H_2(X)\sim H_2(Y)$ via $f_*: H_2(Y)\to H_2(X)$, then f induces a G-homotopy equivalence on the singular sets of the G-actions on Y and X.*

Proof. For $i=0,4$, the hypotheses of (a) imply that $f^*: H^j(C; H^i(X))\to H^j(C; H^i(Y))$ is an isomorphism for all i and j. Hence $f^*: H^4(X)\to H^4(Y)$ is an isomorphism mod p for each $p\mid |G|$, so that $\deg(f)$ is an integer prime to $|G|$. It follows that the Serre spectral sequence of $E_A\times_A(X,Y)\to BA$ collapses for each p-elementary abelian subgroup $A\subseteq G$. It follows from [5, Theorem 1] that f^* induces a $\mathbb{Z}G$-projective stable equivalence $f^*: H^2(X)\sim H^2(Y)$. The details are as follows. First, we may assume that $f_*: H_2(Y)\to H_2(X)$ is surjective. If not, add copies of $G\times S^2$ (freely permuted by G-translation) along free G-orbits of Y. By equivariant obstruction theory, f may be extended to $f': Y'\to X$ such that $f'_*: H_2(Y)\oplus(\mathbb{Z}G)^n\cong H_2(Y')\to H_2(X)$ will be surjective for an appropriate $n\geq 0$. The mapping cone T of f' will be a mod p homology Moore space of homological dimension (mod p) equal to three, and a base point fixed under G. This mapping cone is freely equivalent to the pair (X,Y) as in [5] and $\bar{H}_*(T;\mathbb{Z}_p)=\operatorname{Ker} f'_*$. The hypotheses in (a) implies that $\bar{H}^*(T)$ is cohomologically trivial as a $\mathbb{Z}C$-module for each $C\subseteq G$, $|C|=p$. According to [5, Theorem 1], $\bar{H}^*(T)$ will be $\mathbb{Z}G$-cohomologically trivial, and since it is $\mathbb{Z}$-free, $\bar{H}^*(T)$ is $\mathbb{Z}G$-projective. As in Section 2, we conclude that $H^2(X)\underset{G}{\simeq}H^2(Y')$ and since $H^2(Y')\underset{G}{\simeq}H^2(X)$, we have $H^2(X)\sim H^2(Y)$.

(b) follows from [5] also. $\square$

Next, we determine the structure of the $\mathbb{Z}G$-module $H^2(X)$ according to the singular set of the action, and an additional group cohomology class.

4.7. Proposition. *Let X satisfy (SPC) above. Let X_2 be any two-skeleton for X which is a G-subcomplex. Then:*

(a) *The stable $\mathbb{Z}G$-isomorphism class of $H_2(X_2)$ is well-defined and it depends only on the singular set of the G-action on X. In particular, suppose Y^4 satisfies the same hypotheses as X^4. Then $H_2(Y_2)\sim H_2(X_2)$ if the singular sets $S(X)$ and $S(Y)$ of G-actions on X and Y are G-homotopy equivalent.*

(b) *If G is a p-group, then $H_2(X_2) \sim H_2(Y_2)$ if and only if $S(X)$ and $S(Y)$ are G-homotopy equivalent.*

(c) *There exists a well-defined homology class $\theta(X) \in H_1(G; H_2(X_2))$ such that the pair $(H_2(X_2), \theta(X))$ determine the $\mathbb{Z}G$-module $H_2(X)$ up to the projective stable equivalence.*

(d) *Suppose X and Y are as in* (b) *above such that under the projective stable equivalence $f: H_2(Y_2) \xrightarrow{\sim} H_2(X_2)$ we have $f_*(\theta(Y)) = \theta(X)$. Then $H_2(Y) \sim H_2(X)$.*

Proof. Let X_2 and X_2' be two different 2-skeleta which are G-subcomplexes. In particular, their singular sets coincide. Using the fact that X_2 and X_2' are simply-connected 2-dimensional Moore G-complexes, we find a G-map $f: X_2 \to X_2'$ which extends the identity on the singular sets. Adding free G-cells of dimension 2 to X_2 does not change the stable isomorphism class of f, and in this way, we may assume $f_*: H_2(X_2) \to H_2(X')$ is surjective. Let W be the mapping cone of f. W has a contractible G-singular set, and it is a 3-dimensional Moore space with $\mathbb{Z}$-free homology, hence its homology is stably $\mathbb{Z}G$-free. (To see this, notice that $W/S(W)$ has the same G-homotopy type as W, and $C_*(W/S(W))$ is $\mathbb{Z}G$-free except for a factor $\mathbb{Z}$ in C_0 which accounts for the base point. Therefore $H_3(W/S(W))$ is the only homology group in the $\mathbb{Z}G$-free chain complex $\tilde{C}_*(W/S(W))$, which shows that it is stably $\mathbb{Z}G$-free.) Since in $0 \to H_3(W) \to H_2(X_2) \to H_2(X_2') \to 0$ all modules are $\mathbb{Z}$-free, the exact sequence $0 \to H^2(X_2') \to H^2(X_2) \to H^3(W) \to 0$ is $\mathbb{Z}G$-split, from which (a) follows. The second assertion of (a) is proved by a similar argument.

(b) By adding free G-cells to X and Y, we may assume that $H_2(X_2) \cong H_2(Y_2)$ as $\mathbb{Z}G$-modules. For cyclic subgroups of order p, $X_2^{\mathbb{Z}_p}$ and $Y_2^{\mathbb{Z}_p}$ are determined up to homotopy type by $\hat{H}^*(\mathbb{Z}_p, H^2(X_2)) \cong \hat{H}^*(\mathbb{Z}_p, H^2(Y_2))$ using the localization theorem in equivariant cohomology [19, Chapter 3]. In fact, since $X_2^{\mathbb{Z}_p}$ and $Y_2^{\mathbb{Z}_p}$ are disjoint unions of Riemann surfaces and finite sets of points, $X_2^{\mathbb{Z}_p}$ and $Y_2^{\mathbb{Z}_p}$ are homeomorphic. Taking $\mathbb{Z}_p$ in the center of G, the action of $G/\mathbb{Z}_p$ on $H^*(X_2^{\mathbb{Z}_p})$ and $H^*(Y_2^{\mathbb{Z}_p})$ is also determined by the action of $G/\mathbb{Z}_p$ on $\hat{H}^*(\mathbb{Z}_p, H^2(X_2))$ and $\hat{H}^*(\mathbb{Z}_p, H^2(Y_2))$. In fact, we need to consider only $H^1(\mathbb{Z}_p, H^2(X_2))$ and $H^1(\mathbb{Z}_p, H^2(Y_2))$ as $G/\mathbb{Z}_p$-modules to compute the action of $G/\mathbb{Z}_p$ on $H^1(X_2^{\mathbb{Z}_p})$ and $H^2(Y_2^{\mathbb{Z}_p})$. Without loss of generality, we may assume that $G/\mathbb{Z}_p$ acts effectively on $X_2^{\mathbb{Z}_p}$ and $Y_2^{\mathbb{Z}_p}$. The action of $G/\mathbb{Z}_p$ on $H^*(X_2^{\mathbb{Z}_p})$ and $H^*(Y_2^{\mathbb{Z}_p})$ determines the $(G/\mathbb{Z}_p)$-homotopy types of $X_2^{\mathbb{Z}_p}$ and $Y_2^{\mathbb{Z}_p}$ (using Section 2 and some equivariant obstruction theory). Thus, $X_2^{\mathbb{Z}_p}$ and $Y_2^{\mathbb{Z}_p}$ are $(G/\mathbb{Z}_p)$-homotopy equivalent and hence G-equivalent for each $\mathbb{Z}_p \subset G$. An inductive argument on the lattice of subgroups shows that $S(X_2)$ and $S(Y_2)$ are G-homotopy equivalent.

(c) $C_*(X)$ is a $\mathbb{Z}G$-complex in which $C_3(X)$ and $C_4(X)$ are $\mathbb{Z}G$-free.

We have the exact sequences below:

$$0 \to \mathbb{Z} \to C_4(X) \to C_3(X) \to B_2(X) \to 0,$$

$$\text{(E)} \qquad 0 \to B_2(X) \to H_2(X_2) \to H_2(X) \to 0,$$

$$0 \to H_2(X) \to C_2(X) \to C_1(X) \to C_0(X) \to \mathbb{Z} \to 0.$$

Applying $\mathrm{Hom}_{\mathbb{Z}}(\cdot,\mathbb{Z})$ to the first two sequences, we have:

$$(\mathrm{E}^*)\qquad \begin{aligned} &0\to B_2(X)^*\to C^3(X)\to C^4(X)\to \mathbb{Z}\to 0,\\ &0\to H^2(X)\to H^2(X_2)\to B_2(X)^*\to 0.\end{aligned}$$

Now, $B_2(X)^*$ is a representative in the $\mathbb{Z}G$-projective stable equivalence class of $\omega^2(\mathbb{Z})$. From the second exact sequence of (E^*) we find, up to projective stable equivalence:

$$0\longrightarrow H^2(X)\longrightarrow H^2(X_2)\xrightarrow{\varphi_0}\omega^2(\mathbb{Z})\longrightarrow 0.$$

Applying $\omega^{-2}(\cdot)$, we obtain:

$$0\longrightarrow \omega^{-2}H^2(X)\longrightarrow \omega^{-2}H^2(X_2)\xrightarrow{\varphi}\mathbb{Z}\longrightarrow 0.$$

The class of φ in $\widehat{\mathrm{Hom}}_G(\omega^{-2}H^2(X_2),\mathbb{Z})$ corresponds to the homology class $\theta(X)\in H_1(G;H_2(X_2))$ under the isomorphisms:

$$\begin{aligned}&\widehat{\mathrm{Hom}}_G(\omega^{-2}H^2(X_2),\mathbb{Z})\\ &\quad\cong\hat{H}^0(G,\mathrm{Hom}_{\mathbb{Z}}(\omega^{-2}H^2(X_2),\mathbb{Z}))\\ &\quad\cong\hat{H}^0(G,\omega^2H_2(X_2))\cong\hat{H}^{-2}(G,H_2(X_2))\cong H_1(G,H_2(X_2)).\end{aligned}$$

Since $H_2(X_2)$ is well-defined up to stable equivalence, $H_1(G,H_2(X_2))$ depends on the G-complex X only. On the other hand, the projective stable equivalence class of $\omega^{-2}H^2(X)$, and hence of $H_2(X)$, is completely determined by the stable equivalence class of $H_2(X_2)$ and the class $\theta(X)$. See Section 2.

(d) Follows from (a), (c) and Section 2. □

4.8. Theorem. *Let X^4 be a simply-connected finite Poincaré complex with a regular G-action. Suppose G acts trivially on $H_2(X)$. Then the fixed point sets of all subgroups of G are disjoint unions of 2-spheres and isolated points.*

Proof. By Corollary 4.5, this action cannot be free. Let p be a prime, and $\mathbb{Z}_p\subset G$. Since $X^{\mathbb{Z}_p}$ contains the fixed point sets of larger subgroups, without loss of generality, it suffices to prove the theorem for $G=\mathbb{Z}_p$. Consider the Serre spectral sequence for the Borel fibration: $X\to E_G\times_G X\to BG$ in which $H^i(G;H^j(X))\Rightarrow \mathfrak{G}r(H_G^{i+j}(X))$. Since $X^G\neq\emptyset$, the differentials all vanish. This implies, in particular, that $\theta(X)=0$, so that $H_2(X)$ and $H_2(X_2)$ determine each other up to stable equivalence. More explicitly, let $x_1\in X^G$, and let $X_0=X-\{x_1\}$. Since $\mathbb{Z}_p$ cannot act on such a Poincaré complex X with only one fixed point (see [8], for example), we have $X_0^G\neq\emptyset$. Let $x_0\in X_0^G$, and consider the spectral sequence of the pair $E_G\times_G(X_0,x_0)\to BG$ which has only one row. Therefore, $H_G^*(X_0,x_0)\cong H^*(G,H^2(X_0,x_0))=H^*(G,H^2(X))\cong\bigoplus_{i=1}^l H^*(G;\mathbb{Z})$, where $l=\mathrm{rank}_{\mathbb{Z}}H_2(X)$. Due to the periodicity of group cohomology, we may consider these isomorphisms in sufficiently high dimensions, (in fact dimensions >4 is sufficient). Then we may apply the localization theorem [19, Chapter 3] of Borel directly: $j:(X_0^G,x_0)\to(X_0,x_0)$ induces an isomorphism modulo $H^*(G;\mathbb{Z})$-torsion:

$$j^*:H_G^*(X_0,x_0)\to H_G^*(X_0^G,x_0).$$

In this situation, $H_G^i(X_0^G,x_0)\cong H^i(G,H^2(X))$ vanishes for $i\equiv 1 \pmod 2$, since $H^{2j+1}(\mathbb{Z}_p;\mathbb{Z})=0$ for all $j\geq 0$. Since $H_G^*(X_0^G,x_0)\cong H^*(G;\mathbb{Z})\otimes H^*(X_0^G,x_0)$, all odd dimensional cohomology of (X_0^G,x_0) must vanish. This leads to the desired claim. □

4.9. Corollary. *Let X be a simply-connected Poincaré complex of formal dimension four, $\dim X<\infty$, with a regular G-action. Suppose $H_2(X)\underset{G}{\cong}(\mathbb{Z})^l$, where G acts trivially on $H_2(X)$. Then the fixed set of each non-trivial subgroup of G consists of a disjoint union of 2-spheres and a finite set of points.*

Proof. This follows from Theorem 4.8 and Proposition 4.7. □

4.10. Corollary. *Suppose X is as in Proposition 4.7, and suppose that X^H has a component which is a Riemann surface of positive genus. Then G cannot act trivially on $H_2(X)$.* □

The converse of Theorem 4.8 is also true:

4.11. Corollary. *Let X be as in Proposition 4.7. Suppose that $G=\mathbb{Z}_p$ and X^G consists of 2-dimensional spheres and a finite set of points. Then $H_2(X)\sim\mathbb{Z}^l$ with trivial G-action.*

Proof. As in the proof of Theorem 4.8, we consider the isomorphism in high dimensions $H_G^*(X_0^G,x_0)\cong H^*(G,H^2(X))$. From the hypotheses of the corollary, it follows that $H^i(G,H^2(X))=0$ for $i\equiv 1 \pmod 2$. From the classification of cohomology of $\mathbb{Z}$-free $\mathbb{Z}_p$-modules, we conclude that $H^2(X)$ has indecomposable components which are either isomorphic to $\mathbb{Z}$ or a $\mathbb{Z}G$-projective module. □

4.12. Remark. Taking Euler–Poincaré characteristics as before in $\tilde{G}_0(\mathbb{Z}G)$, we conclude that for finite Poincaré complexes, $H_2(X)$ is stably $\mathbb{Z}G$-isomorphic to $\mathbb{Z}^l$ (for some $l\geq 0$) in Corollary 4.11 above.

4.13. Corollary. *Suppose X is as in Corollary 4.11, and the singular set of X consists of a finite set of points, and $G=\mathbb{Z}_p$. Then $H_2(X)$ is stably isomorphic to $\mathbb{Z}^l$ for some l, with a trivial G-action.* □

The following theorem gives precise information on the relationship between the singular set of G-action on X, and the restriction of the $\mathbb{Z}G$-module $H_2(X)$ to the elements of prime order. We formulate this theorem for $G=\mathbb{Z}_p$.

4.14. Theorem. *Let $G=\mathbb{Z}_p$ act on X, satisfying the hypotheses of Proposition* 4.7, *and suppose that the action is not free. Then $H_2(X)$ is stably isomorphic to the $\mathbb{Z}G$-module $A\oplus B$, where $A\cong\mathbb{Z}^m$, and $B\cong(I_G)^{2m_0}$. Let S be the union of the isolated fixed points and the 2-dimensional spheres in X^G. Then $m_0=\sum g_i$, where g_i are the genera of the surfaces in X^G-S, and m is determined by $\chi(X^G)$ and m_0.*

Proof. Once more, we study the permutation complex $C_*(X)$ of the cellular chains of X. Therefore, $C_*(X^G)$ is a direct summand of $C_*(X)$ and it is a G-subcomplex. As before, $Z_i(\cdot)$, $B_i(\cdot)$, and $H_i(\cdot)$ denote cycles, boundaries and homology modules. Consider X_0 as in the proof of Theorem 4.8 above. We will study the exact sequence of $\mathbb{Z}G$-modules $0\to B_2(X)\to Z_2(X)\to H_2(X)\to 0$ in detail. Consider $C_*(X_0)$ as a subcomplex of $C_*(X)$. From the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Z_2(X_0) \arrow[r] \arrow[d, "j"'] & C_2(X_0) \arrow[r] \arrow[d] & C_1(X_0) \arrow[r] \arrow[d] & \tilde{C}_0(X_0) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_2(X) \arrow[r] & C_2(X) \arrow[r] & C_1(X) \arrow[r] & \tilde{C}_0(X) \arrow[r] & 0
\end{tikzcd}$$

it follows that $j: Z_2(X_0)\to Z_2(X)$ is injective, since the other vertical homomorphisms are inclusions.

4.15. Lemma. *There is a $\mathbb{Z}G$-free module F_0 such that $B_2(X)\cong\mathbb{Z}\oplus F_0$.*

Proof. Since $C_j(X_2)\to C_j(X)$ is injective, we have the commutative diagram below in which the first two rows are exact, and the third row is the sequence of quotients:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & C_4(X_0) \arrow[r] \arrow[d] & C_3(X_0) \arrow[r] \arrow[d] & B_2(X_0) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & C_4(X) \arrow[r] \arrow[d] & C_3(X) \arrow[r] \arrow[d] & B_2(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & C_4(X/X_0) \arrow[r] & C_3(X/X_0) \arrow[r] & B_2(X)/B_2(X_0) \arrow[r] & 0.
\end{tikzcd}$$

Note that $C_i(X/X_0)$ is $\mathbb{Z}G$-free for $i=4,3$, $B_2(X_0)$ is $\mathbb{Z}G$-free, and the third row is also exact. The periodicity of cohomology of $\mathbb{Z}_p$ shows that $B_2(X)/B_2(X_0)\sim\mathbb{Z}$, so that we have the exact sequence below

$$0\to B_2(X_0)\to B_2(X)\to\mathbb{Z}\oplus P\to 0$$

in which P is $\mathbb{Z}G$-projective. Taking Euler-characteristics in $\tilde{G}_0(\mathbb{Z}G)$ and taking into account that X is a finite complex, we find that P is also $\mathbb{Z}G$-free in this case. Since $B_2(X_0)$ is $\mathbb{Z}G$-free and the sequence is $\mathbb{Z}$-split, it is also $\mathbb{Z}G$-split. The lemma follows. $\square$

4.16. Lemma. *$Z_2(X) \cong Z_2(X_0) \oplus \mathbb{Z} \oplus F_0$, where F_0 is $\mathbb{Z}G$-free.*

Proof. Consider the commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B_2(X_0) \arrow[r, shift left, dashed] \arrow[d] & B_2(X) \arrow[l, shift left, dashed] \arrow[r, shift left, dashed] \arrow[d] & F_0 \oplus \mathbb{Z} \arrow[l, shift left, dashed] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_2(X_0) \arrow[r, "j"] \arrow[d] & Z_2(X) \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_2(X_0) \arrow[r, "\cong"] \arrow[d] & H_2(X) \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

The isomorphism $H_2(X_0) \cong H_2(X)$ shows that $Q = Z_2(X)/jZ_2(X_0) \cong F_0 \oplus \mathbb{Z}$. Since the top row is split over $\mathbb{Z}G$ by Lemma 4.15, the assertion of the lemma follows. □

4.17. Lemma. *$Z_2(X)$ is stably $\mathbb{Z}G$-isomorphic to $H_2(X) \oplus \mathbb{Z}$.*

Proof. Since $B_2(X_0)$ is $\mathbb{Z}G$-free, we have

$$Z_2(X_0) \cong B_2(X_0) \oplus H_2(X_0) \cong B_2(X_0) \oplus H_2(X).$$

From Lemma 4.16, it follows that

$$Z_2(X) \cong B_2(X_0) \oplus H_2(X) \oplus \mathbb{Z} \oplus F_0 \cong H_2(X) \oplus \mathbb{Z} \oplus F,$$

where $F = F_0 \oplus B_2(X_0)$ is $\mathbb{Z}G$-free. □

We continue the proof of Theorem 4.14 as follows. Let X_2 be a 2-skeleton of X which is also a G-subcomplex as in Proposition 4.7, so that $H_2(X_2)$ and $Z_2(X)$ are stably $\mathbb{Z}G$-isomorphic. $H_2(X_2)$ may be computed directly from the cellular structure of X_2, which is a 2-dimensional simply-connected Moore G-complex. However, we prefer to use equivariant cohomology which provides a shorter proof. Let $x_0 \in X_2^G = X^G$. $H_G^*(X_2, x_0) \cong H^*(G, H^2(X_2, x_0)) \cong H^*(G, H^2(X_2))$ from the Serre spectral sequence. Decompose $H^2(X_2) = A \oplus B \oplus C$ as a $\mathbb{Z}G$-module where $A = \bigoplus A_i$, $B = \bigoplus B_i$, and C is $\mathbb{Z}G$-projective as follows. A_i are indecomposable and $H^*(G, A_i)$ is non-zero only in even dimensions, and $H^*(G, B_i)$ are non-zero only in odd dimensions. We conclude that $H^*(G, B) \cong \bigoplus_j H_G^{2j+1}(X_2, x)$ and $H^*(G, A) \cong \bigoplus_{j>0} H_G^{2j}(X_2, x)$. In all sufficiently high dimensions, $H_G^j(X_2, x) \cong H_G^j(\Sigma, x)$ by the localization theorem, where Σ is the fixed set [19, Chapter 3]. Therefore,

$$H^j_G(\Sigma,x) \cong H^j(G,\mathbb{Z}) \otimes \tilde{H}^0(\Sigma) \oplus H^{j-1}(G,\mathbb{Z}) \otimes H^1(\Sigma) \oplus H^{j-2}(G,\mathbb{Z}) \otimes H^2(\Sigma).$$

Since $H^j(G,\mathbb{Z})=0$ for $j \equiv 1 \pmod 2$, a counting argument leads to the computation that $H^2(X_2)$ and hence $H_2(X_2)$ have a direct sum decomposition of the form $A \oplus B \oplus C$ where the number of the indecomposable summands B_i in B is equal to the total rank $H_1(\Sigma)=2\sum_i g_i$, and $A=\mathbb{Z}^m$, where $m=\operatorname{rank}(H_0(\Sigma))+\operatorname{rank}(H_2(\Sigma))-1$. Since $Z_2(X)$ is stably $\mathbb{Z}G$-isomorphic to $H_2(X_2)$ by the preceding lemma, we have $H_2(X) \oplus \mathbb{Z} \oplus F \cong A \oplus B \oplus C$, where C is $\mathbb{Z}G$-free. Therefore, $H_2(X) \cong \mathbb{Z}^n \oplus B \oplus F$, where F is $\mathbb{Z}G$-free. Taking Euler characteristic in $G_0(\mathbb{Z}G)$, we find: $(n+2)[\mathbb{Z}]+[B]+[F]=t[\mathbb{Z}G]+s[\mathbb{Z}]$ where $s,t \in \mathbb{Z}$. According to the classification of $\mathbb{Z}[\mathbb{Z}_p]$-modules, $H_2(X) \cong \mathbb{Z}^n \oplus I^{2m_0} \oplus F$, where $m_0=\sum_i g_i$ and F is $\mathbb{Z}G$-free. □

5. Some generalizations

In the previous section we analyzed the relationship between the G-module structure of $H_2(X)$ and the topology of the G-action with emphasis on the case $G=\mathbb{Z}_p$. In this section we illustrate how similar results may be deduced for more general finite group actions on simply-connected 4-manifolds. As before, since the arguments are homological for the most part, we may formulate these results for homology manifolds. Suppose X is a finite simplicial complex of dimension n, and $R=\mathbb{Z}/q\mathbb{Z}$. X is an R-homology n-manifold if the link of each vertex has the same R-homology as S^{n-1}. We say that X has a PL G-action, if for some simplicial subdivision, the elements of G act by simplicial homeomorphisms. It is well-known that in this case, the second barycentric subdivision makes X into a G-CW complex (cf. [11]). If X is a $\mathbb{Z}_p$-homology manifold with a PL $\mathbb{Z}_p$-action, then all the components of $X^{\mathbb{Z}_p}$ are also $\mathbb{Z}_p$-homology manifolds. If X is connected and the action is effective, then $X^{\mathbb{Z}_p}$ is of a strictly smaller dimension. If p is odd, then the codimension of $X^{\mathbb{Z}_p}$ is even. Thus, if X is a $\mathbb{Z}_p$-homology 4-manifold and $p>2$, then $X^{\mathbb{Z}_p}$ has components which are isolated points or 2-dimensional manifolds, since $\mathbb{Z}_p$-homology manifolds of dimension two are manifolds. Now let $|G|=q$, and X be a $\mathbb{Z}_p$-homology 4-manifold with an effective PL G-action. We will call X a regular $\mathbb{Z}_q$-homology G-manifold, if $\dim X^H \leq 2$ for all $H \subseteq G$. Recall that $\operatorname{rank}(G)=\max\{n\colon G$ contains a subgroup isomorphic to $(\mathbb{Z}_p)^n$ for some $p\}$.

5.1. Theorem. *Let X be a regular $\mathbb{Z}_q$-homology G-manifold, where $|G|=q$. Assume that $\pi_1(X)=0$ and $H_2(X) \cong \mathbb{Z}^m \oplus M$ where G acts trivially on $\mathbb{Z}^m$ and M is $\mathbb{Z}G$-projective, $m \geq 0$. Then* $\operatorname{rank}(G) \leq 2$ *and every subgroup $H \cong (\mathbb{Z}_p)^2$ has $X^H \neq \emptyset$.*

Proof. It suffices to consider the case where $G \cong (\mathbb{Z}_p)^n$, $n \geq 2$. Let $C \subset G$, $|C|=p$. Then according to Theorem 4.8 and Corollary 4.9, X^C is a disjoint union of copies of S^2 and isolated points, if X has a regular C-action. But one verifies that the same arguments go through in the above situation as well. Now, G/C acts on X^C and $(X^C)^{G/C} = X^G$.

5.2. Lemma. *G/C acts trivially on $H^*(X^C)$. In particular, G/C acts trivially on $H^0(X^C)$ and $\pi_0(X^C)$.*

Proof. By the localization theorem [19, Chapter 3] for sufficiently large t, $H^t_C(X^C) \cong H^t_C(X)$ (cf. the argument of Theorem 4.8 above). But $H^*_C(X^C) \cong H^*(X^C) \otimes H^*(C)$ and $H^*_C(X) \cong H^*(C, H^2(X)) \oplus Q$, where Q is some graded abelian group. This follows from the spectral sequence of $E_C \times_C X \to BC$ which collapses (cf. Theorem 4.8 and Corollary 4.9). If $m=0$, then X^C is either S^0 or S^2, and it is easy to see that G/C acts trivially on $H^*(X^C)$. If $m>0$, then $H^*(C, H^2(X)) \neq 0$ in positive dimensions, and the hypothesis shows that G/C acts trivially on $H^*(C, H^2(X))$. Moreover, $Q \subset H^*(C) \oplus H^{*+4}(C)$, and it has a trivial (G/C)-action. Therefore, $H^*(X^C) \otimes H^*(C)$ has a trivial (G/C)-action, from which the lemma follows. $\square$

Thus G/C leaves the copies of S^2 and the isolated points invariant. If $x_0 \in X^C$ is an isolated point of X^C, then the isotropy subgroup $G_{x_0} = G$. Since X is a $\mathbb{Z}_p$-homology 4-manifold, we see that $\text{rank}(G) \le 2$ from local considerations (essentially because $\mathbb{Z}_p \times \mathbb{Z}_p$ cannot act freely on a $\mathbb{Z}_p$-homology sphere, and the argument below for the fixed-point free case). If X^C consists of copies of S^2, then G/C acts effectively on S^2, otherwise the G-action on X will not be effective, contrary to the standing hypothesis. Moreover, $(S^2)^{G/C} \neq \emptyset$, and $\text{rank}(G/C) \le 1$. Hence $\text{rank}(G) \le 2$ and $X^G = (X^C)^{G/C} \neq \emptyset$. $\square$

5.3. Corollary. *Suppose that G acts effectively on X, and* $\text{rank}_p(G) > 2$. *Then G cannot act trivially on $H_2(X)$. Moreover,* $\text{rank}_{\mathbb{Z}} H_2(X) \ge 2p-2$. *($X$ satisfies Theorem 5.1 above.)* $\square$

5.4. Theorem. *Let X be a simply-connected regular $\mathbb{Z}_p$-homology G-manifold, where $p \mid |G|$ is a fixed prime. Assume that for all $C \subseteq G$, $|C| = p$, $H_2(X; \mathbb{F}_p)$ is $\mathbb{F}_p C$-projective. Then $H_2(X; \mathbb{F}_p)$ is $\mathbb{F}_p G$-projective. Moreover,* $\text{rank}_p(G) \le 2$, *and if P is any p-subgroup of G, then $X^P \neq \emptyset$.*

Proof. Let G' be a maximal p-elementary abelian subgroup. From the spectral sequence of $E_C \times_C X \to BC$, it follows that X^C is either S^0 or S^2. Since the action preserves the orientation, G'/C must act trivially on $\pi_0(X^C)$ and $(X^C)^{G'/C} \neq \emptyset$. Therefore, $X^{G'} \neq \emptyset$ and $\text{rank}(G') \le 2$. Similarly, if P is any p-subgroup of G, $X^P \neq (X^C)^{P/C} \neq \emptyset$ where C is a central p-order subgroup of P. Now let $x_0 \in X^{G'}$, and let $X_0 = X - \{x_0\}$ as a G'-space. The projectivity assumption on all p-order subgroups implies that $H_2(X_0; \mathbb{F}_p)$ is $\mathbb{F}_p G'$-projective according to Theorem 2.4 of [5] or [2], because X_0 is an $\mathbb{F}_p$-Moore space. Therefore $H_2(X; \mathbb{F}_p) \cong H_2(X_0; \mathbb{F}_p)$ is $\mathbb{F}_p G$-projective. Because this is true for all p-elementary abelian subgroups of G, Chouinard's theorem implies that $H_2(X; \mathbb{F}_p)$ is $\mathbb{F}_p G$-projective. $\square$

5.5. Corollary. *Suppose X is a simply-connected topological 4-manifold with a regular G-action, and assume that $H_2(X)$ is $\mathbb{Z}C$-projective for all cyclic subgroups C of prime order. Then $H_2(X)$ is $\mathbb{Z}G$-projective and* $\operatorname{rank}(G)\leq 2$. *Therefore, $H_2(X)$ is $\mathbb{Z}G$-projective if and only if for each prime order subgroup C,* $\dim_{\mathbb{F}_p} H^*(X^C;\mathbb{F}_p)=2$. □

5.6. Theorem. *Let X have a regular G-action, where G is a p-elementary abelian group, $p>2$. Assume that for each $C\subseteq G$, $|C|=p$, there exists an integer $m\geq 0$ (depending on C) such that $H_2(X)\cong\mathbb{Z}^m\oplus M$, where M is $\mathbb{Z}C$-projective (M depends on C also), and $m<p-2$. Then $H_2(X)\cong\mathbb{Z}^m\oplus M$, where M is $\mathbb{Z}G$-projective, and G acts trivially on $\mathbb{Z}^m$.*

Proof. According to Theorem 4.8 and Corollary 4.9, X^C consists of copies of S^2 and isolated points such that $\operatorname{rank} H^*(X^C)\cong m+2$. Hence $\operatorname{rank} H^0(X^C)<p$ which implies that the (G/C)-action on $H^0(X^C)$ must be trivial. Thus G/C acts effectively on each component S^2 or leaves the isolated points fixed. This shows that $\operatorname{rank}(G/C)\leq 2$, so that $\operatorname{rank}(G)\leq 2$. Moreover, $X^G=(X^C)^{G/C}\neq\emptyset$. Choose $x\in X^C$, and let $X_0=X-\{x\}$. We still have $X_0^G\neq\emptyset$, so choose $x_0\in X_0^G$ (see [8] for example). We proceed to show that $H_2(X)\cong\mathbb{Z}^m\oplus M$, where M is $\mathbb{Z}G$-projective and G acts trivially on $\mathbb{Z}^m$. From the localization theorem [19, Chapter 3] we have that $H_G^*(X_0,x_0)\to H_G^*(X_0^G,x_0)$ is an isomorphism modulo H_G^*-torsion. Therefore, as an H_G^*-module, the H_G^*-rank of $H_G^*(X_0,x_0)$ is equal to H_G^*-rank of $H_G^*(X_0^G,x_0)\cong \operatorname{rank}_{\mathbb{Z}} H^*(X_0^G,x_0)=m$. Thus, the cup product $H^*(G)\otimes H^0(G,H^2(X_0,x_0))\to H^*(G,H^2(X_0,x_0))$ is an isomorphism modulo $H^*(G)$-torsion. Here, we have $H_G^*(X_0,x_0)\cong H^*(G,H^2(X_0,x_0))$. Because $H^2(X_0,x_0)$ is $\mathbb{Z}$-free, and $H^*(G,Z)/\text{Radical}\cong H^*(G;\mathbb{F}_p)/\text{Radical}$ in positive dimensions, we may use $\mathbb{Z}$-coefficients in the localization arguments above. Hence, $H^0(G,H^2(X_0,x_0)\cong\mathbb{Z}^m\oplus B$ where B is $H^*(G)$-torsion. Let the injection $\mathbb{Z}^m\to H^2(X_0,x_0)^G\to H^2(X_0,x_0)$ be $\varphi:\mathbb{Z}^m\to H^2(X_0,x_0)$. Then $\varphi_*:H^*(G,\mathbb{Z}^m)\to H^*(G,H^2(X_0,x_0))$ is an isomorphism modulo $H^*(G)$-torsion. Thus, $M=\operatorname{coker}\varphi$ is $H^*(G)$-torsion. Consider the inclusion $\varrho:(X_0^G,x_0)\to(X_0,x_0)$ which induces the (mod $H^*(G)$-torsion) isomorphism $\varrho^*:H_G^*(X_0,x_0)\to H_G^*(X_0^G,x_0)$. It follows that the induced homomorphism $\varrho^*:H^2(X_0,x_0)\to H^2(X_0^G,x_0)$ is surjective and it may be taken to be the dual to φ, i.e. $\varphi=\varrho_*$ in homology. But, by the hypothesis, for each C we have

$$\begin{tikzcd} H_C^*(X_0^C,x_0) \arrow[r, "\cong"] & H_C^*(X_0^G,x_0) \\ H^*(C,H^2(X_0,x_0)) \arrow[r] \arrow[u, "\cong"] & H^*(C,H^2(X_0^G,x_0)). \arrow[u, "\cong"'] \end{tikzcd}$$

Therefore, $H^*(C,H^2(X_0,x_0))\to H^*(C,H^2(X_0^G,x_0))$ is an $H^*(C)$-isomorphism modulo $H^*(C)$-torsion. But the mapping cone of ϱ is a Moore space, say Y. Moreover,

$$H_C^*(Y,y)\cong H^*(C,H^*(Y,y))\cong H^*(C,\operatorname{coker}\varphi)$$

which vanishes in all sufficiently high dimensions. Here, y is the base point fixed by C corresponding to Y. Therefore, coker φ is cohomologically trivial over G. Since ϱ is a split injection over $\mathbb{Z}$, coker φ is $\mathbb{Z}$-free. By a theorem of Rim, coker φ is $\mathbb{Z}G$-projective. Hence

$$H^2(X) \cong \mathbb{Z}^m \oplus \operatorname{coker} \varphi \cong \mathbb{Z}^m \oplus M$$

where M is $\mathbb{Z}G$-projective and the theorem is proved. $\square$

5.7. Corollary. *Let X and G be as in Theorem 5.6. If for all* prime order *subgroups $C \subsetneq G$, X^C consists of copies of S^2 and isolated points and $|C| > \chi(X^C)$, then for some $m \geq 0$, $H_2(X) \cong \mathbb{Z}^m \oplus M$ where M is $\mathbb{Z}G$-projective and G acts trivially on $\mathbb{Z}^m$. Conversely, if $H_2(X) \cong \mathbb{Z}^m \oplus M$ with $\mathbb{Z}G$-projective M and trivial G-action on $\mathbb{Z}^m$, then X^C consists of copies of S^2 and isolated points with $\chi(X^C) = m+2$.* $\square$

6. Finite domination and finiteness conditions

The main results of this section are Theorem 6.1 and Corollary 6.7 which together with Propositions 4.3 and 4.4 provide necessary and sufficient conditions for a free infinite dimensional G-space X to be homotopy equivalent to a 4-dimensional finite Poincaré complex with a free G-action. These results are necessary steps in the solution of the following general problem: Which homotopy G-actions on a simply-connected closed 4-manifold are equivalent to topological G-actions?

We consider the special case where the G-action on the manifold is expected to be free. See [2] for a discussion of related problems. The method of proof of Theorem 6.1 justifies the remark in Section 4 concerning the passage from $\mathbb{Z}_p$ to a general finite group. For background material, see Section 3 above.

6.1. Theorem. *Let X be an ∞-dimensional free G-space, where G is any finite group. Suppose that non-equivariantly, X is homotopy equivalent to a simply-connected closed* 4*-dimensional Poincaré complex. The following conditions are necessary and sufficient for the existence of a finitely dominated* 4*-dimensional free G-complex Y which is G-homotopy equivalent to X: For each subgroup $C \subsetneq G$, $|C| = p =$ prime,*

(I) *the spectral sequence of the Borel construction $E_C \times_C X \to BC$ does not collapse,*

(II) $\dim_{\mathbb{F}_p} H^1(C; H^2(X)) \geq 2$.

Proof. Suppose X is G-homotopy equivalent to such a Y. Then we may consider $E_C \times_C Y \to BC$ which is fibre homotopy equivalent to $E_C \times_C X \to BC$. Since $E_C \times_C Y \simeq Y/C$, $H^i_C(Y) = 0$ for $i > 4$. From this, it follows that (I) holds. Condition (II) follows from Proposition 4.3 by a direct computation:

$$H^1(C; H^2(Y)) \cong H^1(C; I \oplus I) \cong H^2(C; \mathbb{Z} \oplus \mathbb{Z}) \cong \mathbb{F}_p \oplus \mathbb{F}_p.$$

To prove that these conditions are sufficient, we proceed in several steps. First, we apply our finite domination criterion in [6] to reduce the problem to the case $G=\mathbb{Z}_p$. According to [6, Theorem 3.3], X is G-equivariantly finitely dominated if and only if it is C-equivariantly finitely dominated for each $C\subseteq G$, $|C|=\text{prime}$. See Section 3 above. Thus, we may assume that $G=\mathbb{Z}_p$ from now on. Next, we notice that equivariant finite domination is preserved under 'free equivalence relation' of G-spaces and G-complexes. This translates into the following:

6.2. Lemma. *Let W be a free G-complex obtained from X by adding free orbits of G-cells of dimensions 3 and 4 so that $\pi_i(W)=0$ for $i\leq 3$. Then W is G-finitely dominated if and only if X is G-finitely dominated.*

Proof. This is a special case of [8, Proposition 1.1]. See also [4]. □

Next, we reduce the problem to cohomology computations.

6.3. Lemma. *Let W be as in Lemma 6.2 above. Then W is finitely dominated if and only if $H^i_G(W)=0$ for $i\geq 5$, and this happens if and only if $H^i_G(X)=0$ for $i\geq 5$.*

Proof. Let W' be any free G-complex such that $\pi(W')=0$ for $i\leq 3$ and $\dim W'=4$ (e.g. the universal cover for a 4-skeleton of BG). Since $\pi_i(W)=0$ for $i\leq 3$, we can see by an easy equivariant obstruction theory argument that there exists an equivariant map $f\colon W'\to W$. Assume that $H^i_G(W)=0$ for $i\geq 5$. Compare the Serre spectral sequences of the Borel constructions $E_G\times_G W'\to BG$ and $E_G\times_G W\to BG$. Without loss of generality, we may assume that $f_*\colon H_4(W')\to H_4(W)$ is surjective (otherwise add free orbits of 4-cells and extend f over the new cells to realize geometrically a surjection $(ZG)^l\twoheadrightarrow H_4(W)$). In the cohomology spectral sequences we have:

$$\begin{array}{ccc}
H^j(G,H^4(W))\cong E_5^{j,4} & \xrightarrow{\ f_*^4\ } & E_5'^{j,4}\cong H^j(G,H^4(W')) \\
\Big\downarrow{\scriptstyle d_5} & & \Big\downarrow{\scriptstyle d_5'} \\
H^{j+5}(G)\cong E_5^{j+5,0} & \xrightarrow{\ f_*^0\ } & E_5'^{j+5,0}\cong H^{j+5}(G)
\end{array}$$

in which f_*^0 is an isomorphism, and for all $j>0$, d_5 and d_5' must be also isomorphisms in order to achieve $E_\infty^{i,j}=E_\infty'^{i,j}=0$ for all $i+j\geq 5$. Therefore, f_*^4 must be an isomorphism.

In the long exact sequence of group cohomology associated to the exact sequence

$$0\longrightarrow H^4(W)\xrightarrow{\ f^*\ } H^4(W')\longrightarrow \operatorname{coker}(f^*)\longrightarrow 0,$$

we have that $f_*^4\colon H^j(G,H^4(W))\to H^j(G,H^4(W'))$ is an isomorphism for all but finitely many j. Hence, for all $j\gg 0$, $H^j(G;\ \operatorname{coker} f^*)=0$, which implies that $\operatorname{coker} f^*$ is cohomologically trivial. Since f^* splits over $\mathbb{Z}$, $\operatorname{coker} f^*$ is also $\mathbb{Z}$-torsion free, therefore it is $\mathbb{Z}G$-projective according to Rim's theorem. Consequently, $\operatorname{Ker} f_*$ is $\mathbb{Z}G$-projective. Now we may add free G-cells of dimension 4 and 5 to kill $\operatorname{Ker} f_*$ and extend f to a G-homotopy equivalence $h: W'' \to X$. It is easily seen that W'' is G-finitely dominated and dim $W' \leq 5$, cf. [21]. Also, from the comparison of the spectral sequences of $E_G \times_G X \to BG$ and $E_G \times_G W \to BG$ (or by other elementary arguments) we see that $H_G^i(W) = 0$ for $i \geq 5$ if and only if $H_G^i(X) = 0$ for $i \geq 5$. $\square$

To finish the proof of Theorem 6.1, we only need to show that $H_G^i(X) = 0$ for $i \geq 5$, or equivalently, $E_\infty^{i,j} = 0$ for $i + j \geq 5$ in the Serre spectral sequence mentioned above. This follows from the following lemmas:

6.4. Lemma. *In the spectral sequence of $E_G \times_G X \to BG$ assume that $d_3^{j,2}$ is not identically zero. Then*

(I) $E_\infty^{j,0} = 0$ *for all* $j \geq 3$.

(II) *If* $\zeta \in E_3^{i,2}$ *is such that* $d_3(\zeta) = 0$ *in* $E_3^{i+3,0}$, *then* $\zeta \in \operatorname{Im}(d_3 : E_3^{i-3,4} \to E_3^{i,2})$. *Consequently,* $E_r^{i,2} = 0$ *for* $i \geq 2$ *and* $r \geq 4$.

Proof. Using $\mathbb{Z}$-coefficients, we note that $E_r^{2j+1,0} \cong H^{2j+1}(G) = 0$ for all j. Therefore, there exists $\varphi \in E_3^{2i+1,2} \cong H^{2i+1}(G, H^2(X))$ such that $d_3(\varphi) \in H^{2i+4}(G) \cong \mathbb{Z}_p$ does not vanish. Since $H^*(G)$ is periodic, it follows that d_3 is onto $E_3^{j,0} \cong \mathbb{Z}_p$ for all $j = 2i \geq 4$. In fact, letting $t \in H^2(G)$ be a generator, the equation $d_3(t^l \cdot \varphi) = d_3(t^l) \cdot \varphi + t^l d_3(\varphi) = t^l d_3(\varphi)$ shows that $d_3(\varphi) \neq 0$ if and only if $d_3(t^l \cdot \varphi) \neq 0$, so that we may assume $\varphi \in H^1(G, H^2(X))$ and let $d_3 \varphi = t^2 \in H^4(G)$ for an appropriate choice of t.

(II) Suppose $\zeta' \in H^j(G, H^2(X))$ is such that $d_3(\zeta') = 0$. First, notice that if $j = 2i$, then $d_3(\zeta') \in H^{2i+3}(G) = 0$. But $d_3(\varphi \cdot \zeta') = (d_3 \varphi) \cdot \zeta' + \varphi d_3 \zeta' = t^2 \cdot \zeta' \neq 0$ if $\zeta' \neq 0$. The following argument shows then that ζ' also belongs to the image of d_3. We note that multiplication by $t^{i-1}: H^0(G, H^4(X)) \to H^{2i-2}(G, H^4(X))$ is surjective, because it is the composition

$$H^0(G) \twoheadrightarrow \hat{H}^0(G) \xrightarrow{\cdot t^{i-1}} \hat{H}^{2i-2}(G) \cong H^{2i-2}(G).$$

Therefore, to find $u \in H^{2i-2}(G, H^4(X))$ such that $d_3(u) = \zeta'$ it suffices to let $\zeta' = t^{i-1} \cdot \zeta_0$, and find $v \in H^0(G, H^4(X))$ such that $d_3(v) = \zeta_0$. This is true, because $d_3(\zeta_0) = 0$ if and only if $d_3(\zeta') = t^{i-1} \cdot d_3(\zeta_0) = 0$ and $\zeta_0 = d_3(v)$ if and only if $d_3(t^{i-1} \cdot v) = t^{i-1} d_3(v) = t^{i-1} \cdot \zeta_0 = \zeta'$ (i.e. $u = t^{i-1} \cdot v$). Thus, we are reduced to the following: given $\zeta \in H^1(G, H^2(X))$ such that $d_3(\zeta) = 0$, find $v \in H^2(G, H^4(X))$ such that $d_3(v) = t^2 \cdot \zeta$:

$$\begin{array}{ccc} H^2(G, H^4(X)) & \longrightarrow & H^5(G, H^2(X)) \\ \| & & \| \\ E_3^{2,4} & \xrightarrow{\quad d_3 \quad} & E_3^{5,2} \end{array}$$

Let $\tilde{\varphi}, \tilde{\zeta}: G \to H^2(X)$ be twisted homomorphisms representing φ and ζ respectively, φ as in (I) above. Then we have the pairing $E_3^{1,2} \otimes E_3^{1,2} \xrightarrow{\mu} E_3^{2,4}$ via:

$$H^1(G, H^2(X)) \otimes H^1(G, H^2(X)) \longrightarrow H^2(G, H^2(X) \otimes H^2(X))$$
$$\xrightarrow{\mu_0} H^2(G, H^4(X))$$

in which the first map is group cohomology cup product, and μ_0 is induced by the cup product in X: $H^2(X) \otimes H^2(X) \to H^4(X)$ which is a G-homomorphism when we give the diagonal action to $H^2(X) \otimes H^2(X)$ and the trivial action to $H^4(X)$. (Recall that G preserves the orientation and acts by isometries on $H^2(X)$.) Moreover, d_3 and μ satisfy

$$d_3(\mu_0(\tilde{\zeta} \otimes \tilde{\varphi})) = \mu_0((d_3\tilde{\zeta}) \otimes \tilde{\varphi}) + (-1)^{\deg \zeta} \mu_0(\tilde{\zeta} \otimes d_3 \tilde{\varphi})$$

on the level of cochains, which translates to $d_3(\zeta \cdot \varphi) = (d_3\zeta) \cdot \varphi \pm \zeta \cdot d_3\varphi$ on the level of cohomology, by a slight abuse of notation. Since $d_3\zeta = 0$ and $d_3\varphi = t^2 \neq 0$, it follows that $d_3(\zeta \cdot \varphi) = t^2 \cdot \zeta$ so that $\mu_0(\tilde{\zeta} \otimes \tilde{\varphi})$ represents the desired cohomology class $\upsilon \equiv \zeta \cdot \varphi$. By periodicity of cohomology of G, this finishes the proof of (II). □

6.5.Lemma. *If $d_3 \equiv 0$ on $E_3^{j,2}$, then $H^{2i}(G, H^2(X)) = 0$ for all $i > 0$.*

Proof. Let $\zeta \in H^2(G, H^2(X))$. Then $d_3(\zeta) \in H^5(G) = 0$. Hence $d_3(\varphi \cdot \zeta) = d_3(\varphi) \cdot \zeta \pm \varphi d_3\zeta = \pm(d_3\varphi) \cdot \zeta = \pm t^2 \cdot \zeta$. Since $\varphi \cdot \zeta \in H^3(G, H^4(X)) = 0$, it follows that $t^2 \cdot \zeta = 0$ from which $\zeta = 0$. □

6.6. Lemma. *In the above situation, $E_\infty^{i,j} = 0$ when $i + j > 4$.*

Proof. It remains to show that $E_\infty^{j,4} = 0$ for $j > 0$. By the hypotheses of the theorem, the differential $d_3^{1,2}: H^1(G, H^2(X)) \to H^4(G) \cong \mathbb{Z}_p$ is surjective and has a non-zero kernel. Let $\varphi' \in \operatorname{Ker} d_3^{1,2}$ be non-trivial. Then $d_3(\varphi \cdot \varphi') = (d_3\varphi)\varphi' = t^2 \cdot \varphi' \neq 0$ and $\varphi \cdot \varphi' \in H^2(G, H^4(X))$, where $\varphi \cdot \varphi'$ is the cohomology class represented by $\mu_0(\tilde{\varphi} \otimes \tilde{\varphi}')$ as in Lemma 6.4. Since $H^{2i}(G, H^4(X)) \cong \mathbb{Z}_p$ for all i, it follows that $d_3^{2i,4} \neq 0$ for all $i > 0$ (in fact $i \geq 0$) from which we conclude $E_r^{2i,4} = 0$ for $r \geq 4$. As we noticed before, $E_r^{2i,4} = 0$ for $r \geq 2$. Therefore the lemma is proved. □

6.7. Corollary. *In the situation of Theorem 6.1, if the necessary conditions are satisfied, then there exists a well-defined obstruction $\theta(X) \in \tilde{K}_0(\mathbb{Z}G)$ such that $\theta(X) = 0$ if and only if X is G-homotopy equivalent to a finite Poincaré complex with a free G-action.*

Proof. This follows from the general theory of Wall [21], once one shows that X/G is finitely dominated, which was proved in Theorem 6.1. □

6.8. Remarks. (a) Once the necessary conditions of Theorem 6.1 are satisfied, the obstruction $\theta(X)$ can be determined in terms of the G-module $H^2(X)$, using

Proposition 4.4. Therefore, $\theta(X)$ depends only on the G-module $H^2(X)$. For example, when $G=\mathbb{Z}_p$, Theorem 6.1 and Proposition 4.3 together imply that $H^2(X)\cong I\oplus I\oplus P$ where I is the augmentation ideal and P is $\mathbb{Z}G$-projective. Then $\theta(X)=[P]\in\tilde{K}_0(\mathbb{Z}G)$.

(b) If $\theta(X)=0$, then one may apply Freedman's theory of topological surgery in dimension 4 to decide the remaining necessary conditions for X to be homotopy equivalent to a topological 4-manifold M^4 with a free G-action.

(c) Theorem 6.1 holds in higher dimensions as well, but the details are more involved. This matter will be taken up in a future paper.

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